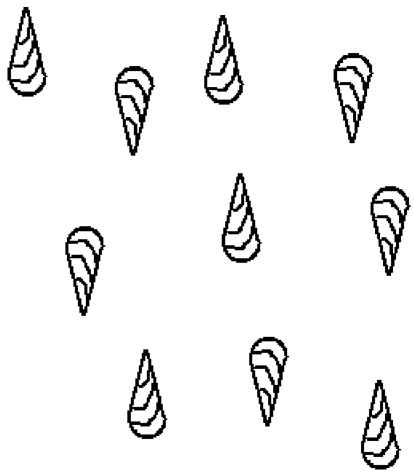
Exercises 04.03.2025 - Symmetry, Miller indices, Tensors - Solutions

1

A non-crystalline medium with symmetry ∞/m must contain an axis of the infinite rotation symmetry perpendicular to a mirror plane. This can be conceived by imagining a medium consisting of parallel screws distributed randomly. One half of the screws is pointing upwards and has right-handed screw thread, whilst the other half is pointing downwards and is left-handed:



2

a) Phase *A* has $m\bar{3}m$ symmetry. The $\langle 011 \rangle$ is the set of all directions that can be obtained from [011] by applying the symmetry operations of the group. Thus, in this phase, there are twelve equivalent crystallographic directions denoted as $\langle 011 \rangle$:

$[011]$, $[01\bar{1}]$, $[101]$, $[10\bar{1}]$, $[110]$, $[1\bar{1}0]$ $[0\bar{1}\bar{1}]$, $[0\bar{1}1]$, $[\bar{1}0\bar{1}]$, $[\bar{1}01]$, $[\bar{1}\bar{1}0]$, and $[\bar{1}10]$.

Phase *B* has *4mm* symmetry. Now there are only four directions that can be obtained from [011] by applying the symmetry operations of the group: $[011]$, $[0\bar{1}1]$, $[\bar{1}01]$, and $[101]$.

b) In phase *A,* the $\{100\}$ is the set of all planes that can be obtained from the plane (100) with the symmetry transformations of $m\bar{3}m$. Thus, there are three crystallographic planes denoted as $\{100\}$: (100), (010), (001).

In phase *B*, there are only two planes obtained from the plane (100) with the symmetry transformations of *4mm*: (100) and (010).

3.

a) The table of direction cosines is defined as $a_{ij} = \cos(X_i', X_j)$:

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

We transform the tensor

$$T_{ij} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

applying the transformation law: $T'_{ij} = a_{i\alpha}a_{j\beta}T_{\alpha\beta}$

$T'_{11} = a_{1\alpha}a_{1\beta}T_{\alpha\beta} = a_{11}^2 T_{11} + a_{12}^2 T_{22} + a_{13}^2 T_{33} = a$

$T'_{12} = a_{1\alpha}a_{2\beta}T_{\alpha\beta} = a_{11}a_{21}T_{11} + a_{12}a_{22}T_{22} + a_{13}a_{23}T_{33} = 0$

$T'_{22} = a_{2\alpha}a_{2\beta}T_{\alpha\beta} = a_{21}^2 T_{11} + a_{22}^2 T_{22} + a_{23}^2 T_{33} = \dfrac{b+c}{2}$

$$T'_{23} = a_{2\alpha}a_{3\beta}T_{\alpha\beta} = a_{21}a_{31}T_{11} + a_{22}a_{32}T_{22} + a_{23}a_{33}T_{33} = \frac{c-b}{2}$$

Having performed all nine calculations of the tensor components, one obtains:

$$T'_{ij} = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{b+c}{2} & \frac{c-b}{2} \\ 0 & \frac{c-b}{2} & \frac{b+c}{2} \end{pmatrix}$$

3b

Use same table of direction cosines for rotation by 45° defined as $a_{ij} = \cos(e'_i, e_j)$:

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

We will use the fact that the tensor transforms equivalently to the vector components product:

$$T_{ij} \sim \begin{pmatrix} p_1q_1 & p_1q_2 & p_1q_3 \\ p_2q_1 & p_2q_2 & p_2q_3 \\ p_3q_1 & p_3q_2 & p_3q_3 \end{pmatrix}$$

With $p'_i = a_{ij}p_j$:

$$p'_1 = p_1$$

$$p'_2 = \frac{1}{\sqrt{2}}(p_2 + p_3)$$

$$p'_3 = \frac{1}{\sqrt{2}}(p_3 - p_2)$$

Using this property, we can transform the tensor T_{ij} as follows:

$$T'_{11} \sim p'_1q'_1 = p_1q_1 \sim T_{11} \Rightarrow T'_{11} = a$$

$$T'_{22} \sim p'_2q'_2 = \frac{1}{2}(p_2 + p_3)(q_2 + q_3) \sim \frac{1}{2}(T_{22} + T_{32} + T_{23} + T_{33}) \Rightarrow T'_{22} = \frac{b+c}{2}$$

$$T'_{23} \sim p'_2q'_3 = \frac{1}{2}(p_2 + p_3)(q_3 - q_2) \sim \frac{1}{2}(T_{23} + T_{33} - T_{22} - T_{32}) \Rightarrow T'_{23} = \frac{c-b}{2}$$

Other six components of the tensor can be found similarly

□

4.

A tensor of fourth-rank is defined by its transformation law. In order to show that c_{ijlk} is a forth rank tensor, we have to show that it transforms as a forth rank tensor under any transition of the reference frames.

The stress and strain tensors transform as a second-rank tensor

$$\sigma'_{ij} = a_{ip} a_{jq} \sigma_{pq} \tag{1}$$

$$\varepsilon'_{ij} = a_{ip} a_{jq} \varepsilon_{pq} \Leftrightarrow \varepsilon_{ij} = a_{pi} a_{qj} \varepsilon'_{pq} \tag{2}$$

In (2) we used that $a_{ij}^{-1} = a_{ji}$

Using (1) and (2) and the Hooke's law, we can rewrite $\sigma_{ij} = c_{ijlk}\varepsilon_{lk}$ as

$$\sigma'_{ij} = a_{ip} a_{jq} c_{pqlk} \varepsilon_{lk} = a_{ip} a_{jq} c_{pqlk} a_{nl} a_{ok} \varepsilon'_{no} = c'_{ijno}\, \varepsilon'_{no}. \tag{3}$$

Where $c'_{ijno} = a_{ip} a_{jq} a_{nl} a_{ok} c_{pqlk}$, showing that c_{ijlk} is a fourth-rank tensor.